Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics
Higher level
Paper 1

Tuesday 12 May 2015 (morning)

Candidate session number

2 hours

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- You are not permitted access to any calculator for this paper.
- Section A: answer all questions in the boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics HL and further mathematics HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[120 marks].**

2215 – 7203

15 pages

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

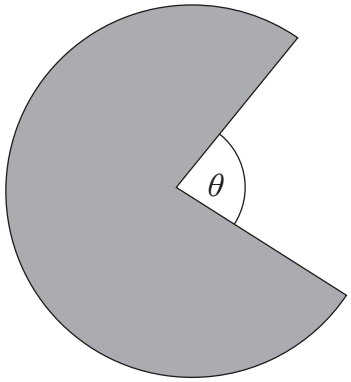
Section A

Answer **all** questions in the boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 5]

The logo, for a company that makes chocolate, is a sector of a circle of radius $2\,\text{cm}$, shown as shaded in the diagram. The area of the logo is $3\pi\,\text{cm}^2$.

diagram not to scale



(a) Find, in radians, the value of the angle θ, as indicated on the diagram. [3]

(b) Find the total length of the perimeter of the logo. [2]

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2. [Maximum mark: 5]

A mathematics test is given to a class of 20 students. One student scores 0, but all the other students score 10.

(a) Find the mean score for the class. [2]

(b) Write down the median score. [1]

(c) Write down the number of students who scored

(i) above the mean score;

(ii) below the median score. [2]

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3. [Maximum mark: 5]

(a) Find $\int (1+\tan^2 x)\,\mathrm{d}x$. [2]

(b) Find $\int \sin^2 x\,\mathrm{d}x$. [3]

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4. [Maximum mark: 5]

(a) Expand $(x+h)^3$. [2]

(b) Hence find the derivative of $f(x)=x^3$ from first principles. [3]

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5. [Maximum mark: 6]

The functions f and g are defined by $f(x) = ax^2 + bx + c$, $x \in \mathbb{R}$ and $g(x) = p\sin x + qx + r$, $x \in \mathbb{R}$ where a, b, c, p, q, r are real constants.

(a) Given that f is an even function, show that $b = 0$. [2]

(b) Given that g is an odd function, find the value of r. [2]

The function h is both odd and even, with domain $\mathbb{R}$.

(c) Find $h(x)$. [2]

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6. [Maximum mark: 7]

A function f is defined by $f(x) = \frac{3x-2}{2x-1}$, $x \in \mathbb{R}$, $x \neq \frac{1}{2}$.

(a) Find an expression for $f^{-1}(x)$. [4]

(b) Given that $f(x)$ can be written in the form $f(x) = A + \frac{B}{2x-1}$, find the values of the constants A and B. [2]

(c) Hence, write down $\int \frac{3x-2}{2x-1} \mathrm{d}x$. [1]

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7. [Maximum mark: 5]

Let $p(x) = 2x^5 + x^4 - 26x^3 - 13x^2 + 72x + 36$, $x \in \mathbb{R}$.

(a) For the polynomial equation $p(x) = 0$, state

(i) the sum of the roots;

(ii) the product of the roots. [3]

A new polynomial is defined by $q(x) = p(x + 4)$.

(b) Find the sum of the roots of the equation $q(x) = 0$. [2]

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8. [Maximum mark: 7]

By using the substitution $u = e^x + 3$, find $\int \frac{e^x}{e^{2x} + 6e^x + 13} dx$.

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9. [Maximum mark: 9]

The functions f and g are defined by $f(x) = 2x + \frac{\pi}{5}$, $x \in \mathbb{R}$ and $g(x) = 3\sin x + 4$, $x \in \mathbb{R}$.

(a) Show that $g \circ f(x) = 3\sin\left(2x + \frac{\pi}{5}\right) + 4$. [1]

(b) Find the range of $g \circ f$. [2]

(c) Given that $g \circ f\left(\frac{3\pi}{20}\right) = 7$, find the next value of x, greater than $\frac{3\pi}{20}$, for which $g \circ f(x) = 7$. [2]

(d) The graph of $y = g \circ f(x)$ can be obtained by applying four transformations to the graph of $y = \sin x$. State what the four transformations represent geometrically and give the order in which they are applied. [4]

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(Question 9 continued)

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10. [Maximum mark: 6]

A football team, Melchester Rovers are playing a tournament of five matches. The probabilities that they win, draw or lose a match are $\frac{1}{2}$, $\frac{1}{6}$ and $\frac{1}{3}$ respectively. These probabilities remain constant; the result of a match is independent of the results of other matches. At the end of the tournament their coach Roy loses his job if they lose three **consecutive** matches, otherwise he does not lose his job. Find the probability that Roy loses his job.

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Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

11. [Maximum mark: 21]

Let $y(x) = xe^{3x}, x \in \mathbb{R}$.

(a) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$. [2]

(b) Prove by induction that $\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = n3^{n-1}\mathrm{e}^{3x} + x3^n\mathrm{e}^{3x}$ for $n \in \mathbb{Z}^+$. [7]

(c) Find the coordinates of any local maximum and minimum points on the graph of $y(x)$. Justify whether any such point is a maximum or a minimum. [5]

(d) Find the coordinates of any points of inflexion on the graph of $y(x)$. Justify whether any such point is a point of inflexion. [5]

(e) Hence sketch the graph of $y(x)$, indicating clearly the points found in parts (c) and (d) and any intercepts with the axes. [2]

Do **not** write solutions on this page.

12. [Maximum mark: 18]

Let $\{u_n\}$, $n \in \mathbb{Z}^+$, be an arithmetic sequence with first term equal to a and common difference of d, where $d \neq 0$. Let another sequence $\{v_n\}$, $n \in \mathbb{Z}^+$, be defined by $v_n = 2^{u_n}$.

(a) (i) Show that $\frac{v_{n+1}}{v_n}$ is a constant.

(ii) Write down the first term of the sequence $\{v_n\}$.

(iii) Write down a formula for v_n in terms of a, d and n. [4]

Let S_n be the sum of the first n terms of the sequence $\{v_n\}$.

(b) (i) Find S_n, in terms of a, d and n.

(ii) Find the values of d for which $\sum_{i=1}^{\infty} v_i$ exists.

You are now told that $\sum_{i=1}^{\infty} v_i$ does exist and is denoted by S_∞.

(iii) Write down S_∞ in terms of a and d.

(iv) Given that $S_\infty = 2^{a+1}$ find the value of d. [8]

Let $\{w_n\}$, $n \in \mathbb{Z}^+$, be a geometric sequence with first term equal to p and common ratio q, where p and q are both greater than zero. Let another sequence $\{z_n\}$ be defined by $z_n = \ln w_n$.

(c) Find $\sum_{i=1}^{n} z_i$ giving your answer in the form $\ln k$ with k in terms of n, p and q. [6]

Do **not** write solutions on this page.

13. [Maximum mark: 21]

Two lines l_1 and l_2 are given respectively by the equations $\boldsymbol{r_1} = \overrightarrow{\mathrm{OA}} + \lambda \boldsymbol{v}$ and $\boldsymbol{r_2} = \overrightarrow{\mathrm{OB}} + \mu \boldsymbol{w}$ where $\overrightarrow{\mathrm{OA}} = \boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}$, $\boldsymbol{v} = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$, $\overrightarrow{\mathrm{OB}} = 2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$, $\boldsymbol{w} = \boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$ and O is the origin. Let P be a point on l_1 and let Q be a point on l_2.

(a) Find $\overrightarrow{\mathrm{PQ}}$, in terms of λ and μ. [2]

(b) Find the value of λ and the value of μ for which $\overrightarrow{\mathrm{PQ}}$ is perpendicular to the direction vectors of both l_1 and l_2. [5]

(c) Hence find the shortest distance between l_1 and l_2. [3]

(d) Find the Cartesian equation of the plane Π, which contains line l_1 and is parallel to the direction vector of line l_2. [5]

Let $\overrightarrow{\mathrm{OT}} = \overrightarrow{\mathrm{OB}} + \eta(\boldsymbol{v} \times \boldsymbol{w})$.

(e) Find the value of η for which the point T lies in the plane Π. [2]

(f) For this value of η, calculate $\left|\overrightarrow{\mathrm{BT}}\right|$. [2]

(g) State what you notice about your answers to (c) and (f), and give a geometrical interpretation of this result. [2]

Please **do not** write on this page.

Answers written on this page
will not be marked.